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Subcritical nonlinear heat equation

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Abstract

We study asymptotic behavior in time of small solutions to nonlinear heat equations in subcritical case. We find a new family of self-similar solutions which change a sign. We show that solutions are stable in the neighborhood of these self-similar solutions.

1. Introduction

This paper is devoted to the study of global existence and large time asymptotic behavior of small solutions to the Cauchy problem for the nonlinear heat equation

$$\begin{cases} u_t - u_{xx} + |u|^\sigma u = 0, & x \in \mathbf{R},\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \end{cases} \tag{1.1}$$

in the subcritical case $\sigma \in (0,2)$.

The blow-up phenomena for positive solutions to the semilinear parabolic equation $u_t - u_{xx} = u^{1+\sigma}$ was obtained in paper [3] for $\sigma \in (0,2)$, in [5] for $\sigma = 2$, and in paper [8] for $\sigma = 2$, for higher space dimensions. Global in time existence of small solutions to (1.1) was proved in [3] for the supercritical case $\sigma > 2$. In the subcritical case $\sigma \in (0,2)$ large time behavior of positive solutions was studied extensively. In paper [4] it was proved that if the initial data are nonnegative

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$u_0 \geqslant 0$, $u_0 \in \mathbf{L}^1$ and decay slowly at infinity as $\lim_{x\to\pm\infty} |x|^{2/\sigma} u_0(x) = +\infty$, then the solution of (1.1) has the asymptotic representation

$$u(t,x) = t^{-\frac{1}{\sigma}} \sigma^{-\frac{1}{\sigma}} + o\big(t^{-\frac{1}{\sigma}}\big)$$

as $t \to \infty$ uniformly in domains $\{x \in \mathbf{R};\ |x| \leqslant C\sqrt{t}\}$ with any $C > 0$. On the other hand in paper [1], there were considered the nonnegative initial data decaying sufficiently rapidly at infinity, i.e. $0 \leqslant u_0(x) \leqslant Ce^{-bx^2}$ for all $x \in \mathbf{R}$, with some $b, C > 0$. Then it was shown that the main term of the asymptotic behavior of solution has a self-similar character

$$u(t,x) = t^{-\frac{1}{\sigma}} w_0\left(\frac{x}{\sqrt{t}}\right) + o\big(t^{-\frac{1}{\sigma}}\big)$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $w_0(\xi)$ is a positive solution of equation

$$-w'' - \frac{\xi}{2} w' + w^{1+\sigma} = \frac{1}{\sigma} w \tag{1.2}$$

which decays rapidly at infinity: $\lim_{|\xi|\to\pm\infty} |\xi|^{2/\sigma} w_0(\xi) = 0$. This result was improved in paper [2], where the intermediate case was considered: if the initial data are such that $u_0 \in \mathbf{L}^1$, $u_0 \neq 0$ and $\lim_{|x|\to\pm\infty} |x|^{2/\sigma} u_0(x) = \varkappa > 0$, then the solutions have the asymptotic representation

$$u(t,x) = t^{-\frac{1}{\sigma}} w_\varkappa\left(\frac{x}{\sqrt{t}}\right) + o\big(t^{-\frac{1}{\sigma}}\big)$$

as $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $w_\varkappa(\xi)$ is a positive solution of Eq. (1.2) such that $\lim_{|\xi|\to\pm\infty} |\xi|^{2/\sigma} w_\varkappa(\xi) = \varkappa$. Note that in these papers there was no restriction on the size of the initial data.

In the present paper we are interested in asymptotic behavior of nonpositive solutions. First we prove the existence of a unique self-similar solution for Eq. (1.1) of the form $(1+t)^{-1/\sigma} S_{\rho,\omega}(\frac{x}{\sqrt{1+t}})$ such that

$$\widehat{S_{\rho,\omega}}(\xi) = \chi_{\rho,\omega}(\xi) + O\big(\varepsilon^{1+\sigma}\big)$$

for all $\xi \in \mathbf{R}$, where $\chi_{\rho,\omega}(\xi) = (\rho + \omega \operatorname{sign} \xi)|\xi|^{2\lambda} e^{-\xi^2}$, $\lambda = \frac{1}{\sigma} - \frac{1}{2}$, and $\varepsilon = |\rho| + |\omega| > 0$ is sufficiently small. Note that these solutions change a sign, since the main part $\mathcal{F}^{-1}_{\xi\to x}\chi_{\rho,\omega}(\xi)$ is nonpositive.

Then in the next theorem we prove the asymptotic stability of these self-similar solutions.

Theorem 1.1. *Let $\frac{4}{3} < \sigma < 2$. We assume that the initial data $u_0 \in \mathbf{H}^{1,1}$ have the mean value $\int_{\mathbf{R}} u_0(x)\,dx = \int_{\mathbf{R}} S_{\rho,\omega}(x)\,dx$ and are close to the self-similar solution $S_{\rho,\omega}$ in the sense*

$$\|u_0 - S_{\rho,\omega}\|_{\mathbf{H}^{1,1}} \leqslant C\varepsilon^{1+\sigma},$$

where $\varepsilon = |\rho| + |\omega| > 0$ *is sufficiently small. Then the Cauchy problem* (1.1) *has a unique global solution* $u(t,x) \in \mathbf{C}([0,\infty); \mathbf{H}^{1,1})$*, satisfying the following time decay estimates*

$$\left\| u(t) - (1+t)^{-\frac{1}{\sigma}} S_{\rho,\omega}\left(\frac{\cdot}{\sqrt{1+t}}\right)\right\|_{\mathbf{L}^\infty} \leqslant C \langle t\rangle^{-\frac{1}{\sigma}-\gamma}$$

for all $t \geqslant 0$*, where* $\gamma > 0$*.*

We considered the one-dimensional case for simplicity, however we believe that our method is also applicable to the multi-dimensional case.

We define the Fourier transformation $\mathcal{F}\phi = \hat{\phi}$ by

$$\hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{-ix\xi} \phi(x)\, dx$$

and $\mathcal{F}^{-1}\phi(x) = \check{\phi}(x)$ is the inverse Fourier transform of ϕ, i.e.

$$\check{\phi}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} e^{ix\xi} \phi(\xi)\, d\xi.$$

We denote by $\mathbf{L}^p$ for $1 \leqslant p \leqslant \infty$, the usual Lebesgue space with a norm $\|\phi\|_{\mathbf{L}^p} = (\int_{\mathbf{R}} |\phi(x)^p\, dx)^{1/p}$ if $1 \leqslant p < \infty$ and $\|\phi\|_{\mathbf{L}^\infty} = \text{ess.sup}_{x\in\mathbf{R}} |\phi(x)|$. Weighted Sobolev space $\mathbf{H}^{m,s} = \{\phi \in \mathbf{L}^2;\ \|\phi\|_{\mathbf{H}^{m,s}} = \|\langle x\rangle^s \langle i\partial_x\rangle^m \phi\|_{\mathbf{L}^2}\} < \infty,\ m, s \geqslant 0$. We denote $\langle x\rangle = \sqrt{1+x^2}$. By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. Different positive constants could be denoted by the same letter C.

We organize our paper as follows. Section 2 is devoted to the proof of preliminary estimates in Lemma 2.1 and to the proof of the existence of self-similar solutions $S_{\rho,\omega}$ in Lemma 2.2. In Section 3 we prove Theorem 1.1.

2. Lemmas

First we give the estimates of the norm $\|\phi\|_{\mathbf{A}^\alpha} = \||\xi|^{-\alpha}\hat{\phi}(\xi)\|_{\mathbf{L}^2}$.

Lemma 2.1. *Let the moments* $\int_{\mathbf{R}} x^j \phi(x)\, dx = 0$ *for* $0 \leqslant j \leqslant n$*. Then the inequality is true*

$$\|\phi\|_{\mathbf{A}^\alpha} \leqslant C\|\phi\|_{\mathbf{L}^2}^{1-\frac{\alpha}{n+\frac{1}{2}+\frac{1}{p}}} \left\||x|^{n+1}\phi\right\|_{\mathbf{L}^p}^{\frac{\alpha}{n+\frac{1}{2}+\frac{1}{p}}}$$

for $\frac{1}{2} < \alpha < n + \frac{1}{2} + \frac{1}{p}$, $1 \leqslant p \leqslant 2$*.*

Proof. Choosing ν such that $\max(\alpha, n+\frac{1}{2}) < \nu < n + \frac{1}{2} + \frac{1}{p}$, and applying the Cauchy–Schwarz inequality with $\delta = \|\phi\|_{\mathbf{L}^2}^{\frac{1}{n+\frac{1}{2}+\frac{1}{p}}} \||x|^{n+1}\phi\|_{\mathbf{L}^p}^{-\frac{1}{n+\frac{1}{2}+\frac{1}{p}}}$, we find

$$\|\phi\|_{\mathbf{A}^\alpha} = \big\||\xi|^{-\alpha}\hat{\phi}(\xi)\big\|_{\mathbf{L}^2}$$
$$\leqslant \left(\int_{\mathbf{R}} dx\,\phi(x)\int_{\mathbf{R}} dy\,\phi(y)\int_{|\xi|\leqslant\delta} d\xi\,|\xi|^{-2\alpha}\left(e^{i\xi x}-\sum_{k=0}^{n}\frac{(i\xi x)^k}{k!}\right)\right.$$
$$\left.\times\left(e^{-i\xi y}-\sum_{k=0}^{n}\frac{(-i\xi y)^k}{k!}\right)\right)^{\frac{1}{2}}+\left(\int_{|\xi|>\delta}|\xi|^{-2\alpha}\big|\hat{\phi}(\xi)\big|^2\,d\xi\right)^{\frac{1}{2}},$$

hence choosing $\mu=\frac{1}{\delta}$, we get

$$\|\phi\|_{\mathbf{A}^\alpha}\leqslant C\int_{|x|\leqslant\mu}\big|\phi(x)\big||x|^{\nu-\frac{1}{2}}\,dx\left(\int_{|\xi|\leqslant\delta}d\xi\,|\xi|^{2\nu-2\alpha-1}\right)^{\frac{1}{2}}$$
$$+C\int_{|x|>\mu}\big|\phi(x)\big||x|^{n+1}\frac{dx}{|x|^{n-\nu+\frac{3}{2}}}\left(\int_{|\xi|\leqslant\delta}d\xi\,|\xi|^{2\nu-2\alpha-1}\right)^{\frac{1}{2}}+C\delta^{-\alpha}\|\phi\|_{\mathbf{L}^2}$$
$$\leqslant C\delta^{\nu-\alpha}\left(\mu^{\nu}\|\phi\|_{\mathbf{L}^2}+\mu^{\nu-n-\frac{1}{2}-\frac{1}{p}}\big\||x|^{n+1}\phi\big\|_{\mathbf{L}^p}\right)+C\delta^{-\alpha}\|\phi\|_{\mathbf{L}^2}$$
$$\leqslant C\|\phi\|_{\mathbf{L}^2}^{1-\frac{\alpha}{n+\frac{1}{2}+\frac{1}{p}}}\big\||x|^{n+1}\phi\big\|_{\mathbf{L}^p}^{\frac{\alpha}{n+\frac{1}{2}+\frac{1}{p}}}.$$

Thus the estimate of the lemma is true. Lemma 2.1 is proved. □

Next we prove the existence of self-similar solutions for Eq. (1.1) of the form $u(t,x)=(1+t)^{-1/\sigma}S(\frac{x}{\sqrt{1+t}})$. By Eq. (1.1) we get the following ordinary differential equation for the function $S(x)$

$$-\frac{1}{2}\frac{d}{dx}(xS)-S''-\lambda S+|S|^\sigma S=0, \tag{2.1}$$

where $\lambda=\frac{1}{\sigma}-\frac{1}{2}$. Denote $\chi_{\rho,\omega}(\xi)=(\rho+\omega\operatorname{sign}\xi)|\xi|^{2\lambda}e^{-\xi^2}$.

Lemma 2.2. *There exists a unique solution of Eq.* (2.1) *such that*

$$\widehat{S_{\rho,\omega}}(\xi)-\chi_{\rho,\omega}(\xi)=O\big(\varepsilon^{1+\sigma}\big) \tag{2.2}$$

for all $\xi\in\mathbf{R}$, *where* $\varepsilon=|\rho|+|\omega|>0$ *is sufficiently small.*

Proof. Applying the Fourier transformation to (2.1) we obtain for $\widehat{S_{\rho,\omega}}(\xi)=\mathcal{F}_{x\to\xi}S_{\rho,\omega}$ the equation (ρ and ω we will omit below)

$$\widehat{S}'+2\xi\widehat{S}=\frac{2}{\xi}\big(\lambda\widehat{S}-\mathcal{F}_{x\to\xi}\big(|S|^\sigma S\big)\big). \tag{2.3}$$

Note that the linear part of Eq. (2.3) has a general solution of the form $(C_1 + C_2 \operatorname{sign}\xi)|\xi|^{2\lambda} e^{-\xi^2}$ with arbitrary constants C_1 and C_2. We look for the solutions in the form

$$\widehat{S} = \chi + \phi + w, \tag{2.4}$$

where

$$\phi(\xi) = \sum_{j=0}^{n} a_j \xi^j e^{-\xi^2},$$

here integer n is such that $n > 2\lambda$. In the case when $2\lambda = l$ is integer, we take a modified representation

$$\phi(\xi) = \sum_{j=0,\, j\neq l}^{n} a_j \xi^j e^{-\xi^2} + a_l \xi^l e^{-\xi^2} \log|\xi|.$$

The constants a_j we will define later by the condition

$$w(\xi) = o\bigl(|\xi|^n\bigr) \tag{2.5}$$

for $\xi \to 0$. Substituting representation (2.4) into (2.3) we find

$$w' + 2\xi w = \frac{1}{\xi}\bigl(2\lambda w - 2\mathcal{F}_{x\to\xi}\bigl(|S|^\sigma S\bigr) + \psi(\xi)\bigr), \tag{2.6}$$

where

$$\psi(\xi) = 2\lambda\phi - 2\xi^2\phi - \xi\phi'.$$

That is we have

$$\psi(\xi) = \sum_{j=0}^{n} a_j (2\lambda - j)\xi^j e^{-\xi^2}$$

and in the case $2\lambda = l$ we obtain

$$\psi(\xi) = \sum_{j=0,\, j\neq l}^{n} (2\lambda - j) a_j \xi^j e^{-\xi^2} - a_l \xi^l e^{-\xi^2}.$$

Now integration of (2.6) with respect to ξ yields

$$w(\xi) = \int_0^{\xi} e^{\eta^2 - \xi^2}\bigl(2\lambda w - 2\mathcal{F}_{x\to\eta}\bigl(|S|^\sigma S\bigr) + \psi(\eta)\bigr)\,\frac{d\eta}{\eta}. \tag{2.7}$$

We write the Taylor expansion

$$2e^{\xi^2}\mathcal{F}_{x\to\xi}\big(|S|^\sigma S\big)=\sum_{j=0}^{n}A_j\xi^j+O\big(\xi^{n+1}\big),$$

where

$$\begin{aligned}A_j&=\frac{1}{j!}\partial_\xi^j\big(2e^{\xi^2}\mathcal{F}_{x\to\xi}\big(|S|^\sigma S\big)\big)\big|_{\xi=0}\\&=2\sum_{l=0}^{j}\frac{1}{(j-l)!l!}\big(\partial_\xi^l e^{\xi^2}\big)\big(\partial_\xi^{l-j}\mathcal{F}_{x\to\xi}\big(|S|^\sigma S\big)\big)\big|_{\xi=0}\\&=(2\pi)^{-\frac{1}{2}}\sum_{k=0}^{[\frac{j}{2}]}\frac{2}{(2k)!(j-2k)!}\int_{\mathbf{R}}(ix)^{j-2k}\big|S(x)\big|^\sigma S(x)\,dx.\end{aligned}$$

Therefore the condition (2.5) implies that

$$a_j=\frac{A_j}{2\lambda-j}\tag{2.8}$$

for $0\leqslant j\leqslant n$, and in the case $2\lambda=l$ we have relations (2.8) for $0\leqslant j\leqslant n$, $j\neq l$, whereas $a_l=-A_l$.

We now solve Eq. (2.7) by the successive approximations. Let $w_0(\xi)=0$, $\phi_{-1}(\xi)=0$ and $w_{m+1}(\xi)$ for $m\geqslant 0$ is defined by the recurrent relations

$$w_{m+1}(\xi)=\int_0^\xi e^{\eta^2-\xi^2}\big(2\lambda w_{m+1}(\eta)-2\mathcal{F}_{x\to\eta}\big(|S_m|^\sigma S_m\big)+\psi_m(\eta)\big)\,\frac{d\eta}{\eta},\tag{2.9}$$

where

$$\widehat{S}_m=\chi+\phi_{m-1}+w_m,$$

$$\phi_{m-1}(\xi)=\sum_{j=0}^{n}a_j^{(m-1)}\xi^j e^{-\xi^2},$$

$$\psi_m(\xi)=\sum_{j=0}^{n}a_j^{(m)}(2\lambda-j)\xi^j e^{-\xi^2},$$

here integer $n>2\lambda$,

$$a_j^{(m)}=\frac{A_j^{(m)}}{2\lambda-j},$$

$$A_j^{(m)} = (2\pi)^{-\frac{1}{2}} \sum_{k=0}^{[\frac{j}{2}]} \frac{2}{(2k)!(j-2k)!} \int_{\mathbf{R}} (ix)^{j-2k} \left|S_m(x)\right|^{\sigma} S_m(x)\,dx$$

for $0 \leqslant j \leqslant n$, whereas in the case when $2\lambda = l$ is integer, we have

$$\phi_{m-1}(\xi) = \sum_{j=0,\, j\neq l}^{n} a_j^{(m-1)} \xi^j e^{-\xi^2} + a_l^{(m-1)} \xi^l e^{-\xi^2} \log|\xi|,$$

$$\psi_m(\xi) = \sum_{j=0,\, j\neq l}^{n} (2\lambda - j) a_j^{(m)} \xi^j e^{-\xi^2} - a_l^{(m)} \xi^l e^{-\xi^2},$$

and relations

$$a_j^{(m)} = \frac{A_j^{(m)}}{2\lambda - j}$$

for $0 \leqslant j \leqslant n$, $j \neq l$, and $a_l^{(m)} = -A_l^{(m)}$, $l = 2\lambda$.

Let us prove the estimates

$$\left\| |\xi|^{-n} w_m(\xi) \right\|_{\mathbf{L}^2} \leqslant C_1 \varepsilon^{1+\sigma}, \tag{2.10}$$

$$\left\| \langle\xi\rangle w_m(\xi) \right\|_{\mathbf{L}^2} \leqslant C_2 \varepsilon^{1+\sigma}, \tag{2.11}$$

$$\left\| \langle\xi\rangle \partial_\xi^n w_m \right\|_{\mathbf{L}^2} \leqslant C_2 \varepsilon^{1+\sigma} \tag{2.12}$$

and

$$\sum_{j=0}^{n} \left|a_j^{(m)}\right| \leqslant C_3 \varepsilon^{1+\sigma} \tag{2.13}$$

with $n > \max(2\lambda + \frac{1}{2}, \frac{1}{\sigma})$.

For $m = 0$ estimates (2.10)–(2.13) are true. Then by induction we suppose that these estimates are valid for some $m > 0$.

Consider the first estimate (2.10). Note that changing the order of integration we get

$$\begin{aligned}
\left\| |\xi|^{-n} \int_0^{\xi} \Phi(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2}^2 &= \int_{\mathbf{R}} d\xi\, |\xi|^{-2n} \int_0^{\xi} \overline{\Phi(\zeta)} \frac{d\zeta}{\zeta} \int_0^{\xi} \Phi(\eta) \frac{d\eta}{\eta} \\
&= \int_0^{\infty} \frac{d\zeta}{\zeta} \overline{\Phi(\zeta)} \int_{\zeta}^{\infty} d\xi\, |\xi|^{-2n} \int_0^{\zeta} \Phi(\eta) \frac{d\eta}{\eta} \\
&\quad + \int_0^{\infty} \frac{d\eta}{\eta} \Phi(\eta) \int_{\eta}^{\infty} d\xi\, |\xi|^{-2n} \int_0^{\eta} \frac{d\zeta}{\zeta} \overline{\Phi(\zeta)}
\end{aligned}$$

$$+\int_{-\infty}^{0}\frac{d\zeta}{\zeta}\overline{\Phi(\zeta)}\int_{-\infty}^{\zeta}d\xi|\xi|^{-2n}\int_{\zeta}^{0}\Phi(\eta)\frac{d\eta}{\eta}$$

$$+\int_{-\infty}^{0}\frac{d\eta}{\eta}\Phi(\eta)\int_{-\infty}^{\eta}d\xi|\xi|^{-2n}\int_{\eta}^{0}\frac{d\zeta}{\zeta}\overline{\Phi(\zeta)},$$

then applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}\left\||\xi|^{-n}\int_0^{\xi}\Phi(\eta)\frac{d\eta}{\eta}\right\|_{\mathbf{L}^2}^2 &\leqslant \frac{2}{2n-1}\int_{\mathbf{R}}d\zeta\left|\Phi(\zeta)\right||\zeta|^{-2n}\left|\int_0^{\zeta}\Phi(\eta)\frac{d\eta}{\eta}\right|\\ &\leqslant \frac{2}{2n-1}\left\||\zeta|^{-n}\Phi\right\|_{\mathbf{L}^2}\left\||\zeta|^{-n}\int_0^{\zeta}\Phi(\eta)\frac{d\eta}{\eta}\right\|_{\mathbf{L}^2},\end{aligned}$$

hence the inequality follows

$$\left\||\xi|^{-n}\int_0^{\xi}\Phi(\eta)\frac{d\eta}{\eta}\right\|_{\mathbf{L}^2}\leqslant\frac{2}{2n-1}\left\||\xi|^{-n}\Phi\right\|_{\mathbf{L}^2}. \tag{2.14}$$

Applying estimate (2.14) to Eq. (2.9) we get

$$\begin{aligned}&\left\||\xi|^{-n}w_{m+1}\right\|_{\mathbf{L}^2}\\ &\quad\leqslant 2\lambda\left\||\xi|^{-n}\int_0^{\xi}e^{\eta^2-\xi^2}w_{m+1}(\eta)\frac{d\eta}{\eta}\right\|_{\mathbf{L}^2}\\ &\qquad+2\left\||\xi|^{-n}\int_0^{\xi}e^{\eta^2-\xi^2}\left(\mathcal{F}_{x\to\eta}\left(|S_m|^{\sigma}S_m\right)-\psi_m(\eta)\right)\frac{d\eta}{\eta}\right\|_{\mathbf{L}^2}\\ &\quad\leqslant\frac{4\lambda}{2n-1}\left\||\xi|^{-n}w_{m+1}\right\|_{\mathbf{L}^2}+\frac{2}{2n-1}\left\||\xi|^{-n}\left(\mathcal{F}_{x\to\xi}\left(|S_m|^{\sigma}S_m\right)-\psi_m\right)\right\|_{\mathbf{L}^2}.\end{aligned}$$

We can choose $1\leqslant p\leqslant 2$ and $n>2\lambda+\frac{1}{2}$ such that $\frac{p+2}{2pn+p-2}<\sigma$, hence we have

$$\left\||x|^{\frac{n+1}{1+\sigma}}S_m\right\|_{\mathbf{L}^p}\leqslant C\left\|\langle x\rangle^n S_m\right\|_{\mathbf{L}^2}.$$

Therefore by the condition $2n-1>4\lambda$ and applying Lemma 2.1, we obtain

$$\begin{aligned}\left\||\xi|^{-n}w_{m+1}\right\|_{\mathbf{L}^2}&\leqslant C\left\||\xi|^{-n}\left(\mathcal{F}_{x\to\xi}\left(|S_m|^{\sigma}S_m\right)-\psi_m\right)\right\|_{\mathbf{L}^2}\\ &\leqslant C\left\|x^{n+1}|S_m|^{\sigma}S_m\right\|_{\mathbf{L}^p}+C\left\||S_m|^{\sigma}S_m\right\|_{\mathbf{L}^2}\end{aligned}$$

$$
\begin{aligned}
&= C \left\| \left| |x|^{\frac{n+1}{1+\sigma}} S_m \right|^{\sigma} |x|^{\frac{n+1}{1+\sigma}} S_m \right\|_{\mathbf{L}^p} + C \|S_m\|_{\mathbf{L}^2} \|S_m\|_{\mathbf{L}^\infty}^{\sigma} \\
&\leqslant C \left\| \langle x \rangle^n S_m \right\|_{\mathbf{L}^2} \left\| \langle x \rangle^n S_m \right\|_{\mathbf{L}^\infty}^{\sigma} \leqslant C_1 \varepsilon^{1+\sigma}.
\end{aligned}
$$

Thus estimate (2.10) is fulfilled with m replaced by $m+1$.

Now we prove estimate (2.11). In view of inequality $e^{\eta^2-\xi^2} \leqslant e^{-(\xi-\eta)^2}$ for $0 \leqslant \frac{\eta}{\xi} \leqslant 1$ we have by the Hölder inequality

$$
\begin{aligned}
\left\| \langle \xi \rangle \int_0^{\xi} e^{\eta^2-\xi^2} \Phi(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} &\leqslant C \left\| |\xi|^{-n} \Phi \right\|_{\mathbf{L}^2} + C \|\Phi\|_{\mathbf{L}^2} \leqslant C \left\| |\xi|^{-n} \Phi \right\|_{\mathbf{L}^2} \\
&\quad + C \left\| |\xi|^{-n} \Phi \right\|_{\mathbf{L}^2}^{\frac{1}{1+n}} \left\| \langle \xi \rangle \Phi \right\|_{\mathbf{L}^2}^{\frac{n}{1+n}}.
\end{aligned}
$$

Then from Eq. (2.9) we find

$$
\begin{aligned}
&\left\| \langle \xi \rangle w_{m+1} \right\|_{\mathbf{L}^2} \\
&\quad \leqslant 2\lambda \left\| \langle \xi \rangle \int_0^{\xi} e^{\eta^2-\xi^2} w_{m+1}(\eta) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} \\
&\qquad + \left\| \langle \xi \rangle \int_0^{\xi} e^{\eta^2-\xi^2} \left(2\mathcal{F}_{x\to\eta}\left(|S_m|^{\sigma} S_m \right) - \psi_m(\eta) \right) \frac{d\eta}{\eta} \right\|_{\mathbf{L}^2} \\
&\quad \leqslant \frac{1}{2} \left\| \langle \xi \rangle w_{m+1} \right\|_{\mathbf{L}^2} + C \left\| |\xi|^{-n} w_{m+1} \right\|_{\mathbf{L}^2} + C \left\| \langle x \rangle^n S_m \right\|_{\mathbf{L}^2} \|S_m\|_{\mathbf{L}^\infty}^{\sigma} \\
&\quad \leqslant \frac{1}{2} \left\| \langle \xi \rangle w_{m+1} \right\|_{\mathbf{L}^2} + C\varepsilon^{1+\sigma}.
\end{aligned}
$$

Thus we have estimate (2.11) with m replaced by $m+1$.

Now we consider estimate (2.12). We differentiate n times Eq. (2.9) to find

$$
\begin{aligned}
&\frac{d^n}{d\xi^n} w_{m+1}(\xi) + 2 \frac{d^{n-1}}{d\xi^{n-1}} \left(\xi w_{m+1}(\xi) \right) \\
&\quad = 2\lambda \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} w_{m+1} \right) - \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} \left(2\mathcal{F}_{x\to\xi}\left(|S_m|^{\sigma} S_m \right) - \psi_m \right) \right).
\end{aligned}
$$

Multiplying the last equation by $\frac{d^n}{d\xi^n} w_{m+1}(\xi)$ and integrating the result over $\xi \in \mathbf{R}$ we get

$$
\begin{aligned}
\left\| \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} &\leqslant C \left\| \frac{d^{n-2}}{d\xi^{n-2}} w_{m+1} \right\|_{\mathbf{L}^2} + 2\lambda \left\| \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} w_{m+1} \right) \right\|_{\mathbf{L}^2} \\
&\quad + \left\| \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} \left(2\mathcal{F}_{x\to\xi}\left(|S_m|^{\sigma} S_m \right) - \psi_m \right) \right) \right\|_{\mathbf{L}^2}.
\end{aligned}
$$

By interpolation we have

$$\left\| \frac{d^{n-2}}{d\xi^{n-2}} w_{m+1} \right\|_{\mathbf{L}^2} \leqslant \frac{1}{3} \left\| \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} + C \|w_{m+1}\|_{\mathbf{L}^2}$$

and

$$\left\| \frac{d^{n-1}}{d\xi^{n-1}} \left(\frac{1}{\xi} w_{m+1} \right) \right\|_{\mathbf{L}^2} \leqslant \frac{1}{3} \left\| \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} + C \left\| |\xi|^{-n} w_{m+1} \right\|_{\mathbf{L}^2} + C \|w_{m+1}\|_{\mathbf{L}^2},$$

hence we obtain

$$\left\| \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} \leqslant C_3 \varepsilon^{1+\sigma}.$$

In the same manner we get the estimate

$$\left\| \xi \frac{d^n}{d\xi^n} w_{m+1} \right\|_{\mathbf{L}^2} \leqslant C_3 \varepsilon^{1+\sigma}.$$

Thus estimate (2.12) with m replaced by $m+1$ is valid. The rest estimate (2.13) follows from the previous estimates via formulas

$$\begin{aligned}
\sum_{j=0}^{n} \left| a_j^{(m+1)} \right| &\leqslant C \int_{\mathbf{R}} \langle x \rangle^n \left| S_{m+1}(x) \right|^{\sigma+1} dx \\
&\leqslant C \left\| \langle x \rangle^n S_{m+1} \right\|_{\mathbf{L}^2} \left\| \langle x \rangle^n S_{m+1} \right\|_{\mathbf{L}^\infty}^{\sigma} \\
&\leqslant C \left\| \langle \xi \rangle \langle \partial_\xi \rangle^n (\chi + \phi_m + w_{m+1}) \right\|_{\mathbf{L}^2}^{1+\sigma} \leqslant C_3 \varepsilon^{1+\sigma}.
\end{aligned}$$

Therefore estimate (2.13) follows with m replaced by $m+1$. Thus by induction estimates (2.10)–(2.13) are true for all m.

In the same manner we prove the estimate

$$\|w_{m+1} - w_m\|_{\mathbf{X}} \leqslant \frac{1}{2} \|w_m - w_{m-1}\|_{\mathbf{X}},$$

where $\|\psi\|_{\mathbf{X}} \equiv \| |\xi|^{-n} \psi \|_{\mathbf{L}^2} + \| \langle \xi \rangle \psi \|_{\mathbf{L}^2} + \| \langle \xi \rangle \partial_\xi^n \psi \|_{\mathbf{L}^2}$. Therefore there exists a unique solution of Eq. (2.1) having the form (2.2). Lemma 2.2 is proved. □

3. Proof of Theorem 1.1

The local existence of solutions for the Cauchy problem (1.1) can be obtained by the standard contraction mapping principle. In order to get a priori estimates of local solutions as in the papers [6,7] we make a change of the dependent variable $u(t,x) = e^{-\varphi(t)} v(t,x)$, then we get

$$v_t - v_{xx} + e^{-\sigma\varphi} |v|^\sigma v - v\varphi' = 0. \tag{3.1}$$

We choose $\varphi(t)$ by the condition

$$\int_{\mathbf{R}} \left(e^{-\sigma\varphi}|v|^{\sigma} v - \varphi'(t)v\right) dx = 0,$$

where the initial value $\varphi(0) = 0$. Then the mean value of v satisfies the conservation law

$$\frac{d}{dt}\int_{\mathbf{R}} v(t,x)\,dx = 0,$$

hence

$$\int_{\mathbf{R}} v(t,x)\,dx = \int_{\mathbf{R}} v(0,x)\,dx = \theta = \int_{\mathbf{R}} u_0(x)\,dx.$$

Thus we obtain from (3.1)

$$\begin{cases} v_t - v_{xx} = e^{-\sigma\varphi}\left(\dfrac{v}{\theta}\displaystyle\int_{\mathbf{R}} |v|^{\sigma} v\,dx - |v|^{\sigma} v\right), \\ \varphi'(t) = \dfrac{1}{\theta} e^{-\sigma\varphi}\displaystyle\int_{\mathbf{R}} |v|^{\sigma} v\,dx. \end{cases} \tag{3.2}$$

Now we substitute $v = (1+t)^{\lambda} f + w$, where f is a self-similar solution such that $f = (1+t)^{-1/\sigma} S(\frac{x}{\sqrt{1+t}})$, of Eq. (1.1), $\lambda = \frac{1}{\sigma} - \frac{1}{2}$. By Eq. (1.1) we see that $(1+t)^{\lambda} f$ satisfies

$$\begin{aligned} \left((1+t)^{\lambda} f\right)_t = {} & \left((1+t)^{\lambda} f\right)_{xx} + \lambda(1+t)^{\lambda-1} f \\ & - (1+t)^{-\sigma\lambda}\left|(1+t)^{\lambda} f\right|^{\sigma} (1+t)^{\lambda} f, \end{aligned}$$

then we get for w

$$\begin{aligned} w_t = {} & w_{xx} + e^{-\sigma\varphi}\left(\frac{v}{\theta}\int_{\mathbf{R}} |v|^{\sigma} v\,dx - |v|^{\sigma} v\right) \\ & - \lambda(1+t)^{\lambda-1} f + (1+t)^{-\sigma\lambda}\left|(1+t)^{\lambda} f\right|^{\sigma} (1+t)^{\lambda} f. \end{aligned}$$

Note that the mean value is conserved

$$\int_{\mathbf{R}} v(t,x)\,dx = \int_{\mathbf{R}} \frac{1}{\sqrt{1+t}} S\left(\frac{x}{\sqrt{1+t}}\right) dx + \int_{\mathbf{R}} w(t,x)\,dx = \theta.$$

If we choose the mean value θ, such that $\int_{\mathbf{R}} S(x)\,dx = \theta$, hence we obtain

$$\int_{\mathbf{R}} w(t,x)\,dx = 0.$$

Denote $h(t) = e^{\sigma\varphi(t)}$. We now prove existence of the solution $(w(t,x), h(t))$ by the successive approximations. Let $w_0 = 0$, $h_0 = (1+t)^{1-\sigma/2}$ and $(w_m(t,x), h_m(t))$, for $m = 1, 2, \dots$ we define by equations

$$\partial_t w_m - \partial_x^2 w_m = h_{m-1}^{-1}(t)\left(\frac{v_m}{\theta}\int_{\mathbf{R}} |v_{m-1}|^\sigma v_{m-1}\,dx - |v_{m-1}|^\sigma v_{m-1}\right) \\ - \lambda(1+t)^{\lambda-1} f + (1+t)^{-\sigma\lambda}\left|(1+t)^\lambda f\right|^\sigma (1+t)^\lambda f, \tag{3.3}$$

and

$$h'_m(t) = \frac{\sigma}{\theta}\int_{\mathbf{R}} |v_m|^\sigma v_m\,dx. \tag{3.4}$$

Applying the Fourier transformation to (3.3) and changing the variables $\widehat{w}_m(t,\xi) = z_m(t,\eta)$, $\eta = \xi\sqrt{t+1}$, we get

$$\partial_t z_m + \frac{\eta}{2(t+1)}\partial_\eta z_m + \frac{\eta^2}{t+1} z_m = \psi_{m-1} z_m - g_{m-1}, \tag{3.5}$$

where we denote

$$\psi_{m-1}(t) = \frac{1}{\theta h_{m-1}(t)}\int_{\mathbf{R}} |v_{m-1}|^\sigma v_{m-1}(t,x)\,dx,$$

$$g_{m-1}(t,\eta) = \frac{1}{h_{m-1}(t)}\mathcal{F}_{x\to\eta(1+t)^{-\frac{1}{2}}}\left(|v_{m-1}|^\sigma v_{m-1}\right) \\ - (1+t)^{-1}\mathcal{F}_{x\to\eta}\left(|S|^\sigma S\right) + \left(\psi_{m-1}(t) - \lambda(1+t)^{-1}\right)\widehat{S}.$$

Let us prove the estimates

$$\left\|\eta^{-1} z_m\right\|_{\mathbf{L}^2} + \|z_m\|_{\mathbf{H}^{1,1}} \leqslant C\varepsilon^{1+\sigma}(1+t)^{-\gamma} \tag{3.6}$$

and

$$\left|h_{m-1}(t) - (1+t)^{1-\frac{\sigma}{2}}\right| \leqslant C\varepsilon^\sigma(1+t)^{1-\frac{\sigma}{2}-\gamma}, \tag{3.7}$$

where $\gamma > 0$ is sufficiently small, and $\frac{1}{2} > \lambda + \varepsilon$, $\lambda = \frac{1}{\sigma} - \frac{1}{2}$. By induction we suppose that (3.6)–(3.7) are fulfilled for some m.

Then we have the estimate

$$h'_m(t) = \frac{\sigma}{\theta}\int_{\mathbf{R}} |v_m|^\sigma v_m\,dx \\ = \frac{\sigma}{\theta}\int_{\mathbf{R}} \left|(1+t)^\lambda f + w_m\right|^\sigma \left((1+t)^\lambda f + w_m\right)dx$$

$$= \frac{\sigma}{\theta}(1+t)^{-\frac{\sigma}{2}} \int_{\mathbf{R}} |S|^{\sigma} S\,dx + O\big(\varepsilon^{\sigma}(1+t)^{-\frac{\sigma}{2}-\gamma}\big)$$
$$= \lambda\sigma(1+t)^{-\frac{\sigma}{2}} + O\big(\varepsilon^{\sigma}(1+t)^{-\frac{\sigma}{2}-\gamma}\big)$$

since $\int_{\mathbf{R}} |S|^{\sigma} S\,dx = \lambda\theta$. Integrating this estimate with respect to time we get estimate (3.7) with $m-1$ replaced by m. In the same manner we have

$$\psi_m(t) = \frac{1}{\theta h_m(t)} \int_{\mathbf{R}} |v_m|^{\sigma} v_m\,dx = \lambda(1+t)^{-1}\big(1 + O\big(\varepsilon^{\sigma}(1+t)^{-\gamma}\big)\big),$$

in particular

$$\psi_m(t) \leqslant \frac{\lambda+\varepsilon^{\sigma}}{1+t}.$$

We multiply Eq. (3.5) by $\overline{z_{m+1}}\eta^{-2}$ and integrate the result over $\eta \in \mathbf{R}$ to find

$$\frac{d}{dt}\big\|\eta^{-1}z_{m+1}\big\|_{\mathbf{L}^2}^2 + \frac{1}{2(t+1)} \int_{\mathbf{R}} \eta^{-1}\partial_\eta |z_{m+1}|^2\,d\eta + \frac{2}{t+1}\|z_{m+1}\|_{\mathbf{L}^2}^2$$
$$\leqslant \frac{2\lambda+2\varepsilon^{\sigma}}{1+t}\big\|\eta^{-1}z_{m+1}\big\|_{\mathbf{L}^2}^2 - 2\,\mathrm{Re}\int_{\mathbf{R}} g_{m-1}|\eta|^{-2}\overline{z_{m+1}}\,d\eta. \tag{3.8}$$

Integrating by parts we get

$$\int_{\mathbf{R}} \eta^{-1}\partial_\eta |z_{m+1}|^2\,d\eta = \big\|\eta^{-1}z_{m+1}\big\|_{\mathbf{L}^2}^2.$$

Using Lemma 2.1 and taking into account estimates (3.6)–(3.7) we obtain

$$\big\|\eta^{-1}g_m\big\|_{\mathbf{L}^2} \leqslant C\varepsilon^{1+\sigma}(1+t)^{-1-\gamma},$$

hence (3.8) gives

$$\frac{d}{dt}\big\|\eta^{-1}z_{m+1}\big\|_{\mathbf{L}^2} \leqslant -\frac{\varepsilon^{\sigma}}{t+1}\big\|\eta^{-1}z_{m+1}\big\|_{\mathbf{L}^2} + C\varepsilon^{1+\sigma}(1+t)^{-1-\gamma} \tag{3.9}$$

since $2\lambda + 2\varepsilon^{\sigma} - \frac{1}{2} \leqslant -\varepsilon^{\sigma}$ which follows from $\sigma > \frac{4}{3}$. Integration of (3.9) with respect to time yields the estimate

$$\big\||\eta|^{-1}z_{m+1}\big\|_{\mathbf{L}^2} \leqslant C\varepsilon^{1+\sigma}(1+t)^{-\gamma}. \tag{3.10}$$

Now let us prove the estimate $\|\langle\eta\rangle z_{m+1}\|_{\mathbf{L}^2} \leqslant C\varepsilon^{1+\sigma}(1+t)^{-\gamma}$. We multiply (3.5) by $\langle\eta\rangle^2\overline{z_{m+1}}$ and integrate over $\eta \in \mathbf{R}$ to find

$$\frac{d}{dt}\left\|\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2+\frac{1}{2(t+1)}\int_{\mathbf{R}}\eta\langle\eta\rangle^2\partial_\eta|z_{m+1}|^2\,d\eta+\frac{2}{t+1}\left\|\eta\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2$$
$$=2\psi_m\left\|\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2-2\operatorname{Re}\int_{\mathbf{R}}\langle\eta\rangle^2 g_m\overline{z_{m+1}}\,d\eta.$$

Integration by parts yields

$$-\int_{\mathbf{R}}\eta\langle\eta\rangle^2\partial_\eta|z_{m+1}|^2\,d\eta\leqslant 3\left\|\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2.$$

Therefore using estimate

$$\left\|\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2\leqslant\left\|\eta\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2+C\left\|\eta^{-1}z_{m+1}\right\|_{\mathbf{L}^2}^2\leqslant\left\|\eta\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2+C\varepsilon^{2+2\sigma}(1+t)^{-2-2\gamma}$$

we find

$$\frac{d}{dt}\left\|\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2\leqslant-\frac{2}{t+1}\left\|\eta\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2+\frac{C}{t+1}\left\|\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2+C\varepsilon^{2+2\sigma}(1+t)^{-2-2\gamma}$$
$$\leqslant-\frac{\varepsilon^\sigma}{t+1}\left\|\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}^2+C\varepsilon^{2+2\sigma}(1+t)^{-2-2\gamma}. \tag{3.11}$$

Integration of (3.11) with respect to time yields the estimate

$$\left\|\langle\eta\rangle z_{m+1}\right\|_{\mathbf{L}^2}\leqslant C\varepsilon^{1+\sigma}(1+t)^{-1-\gamma}.$$

In the same way we obtain the estimate

$$\left\|\langle\eta\rangle\partial_\eta z_{m+1}\right\|_{\mathbf{L}^2}\leqslant C\varepsilon^{1+\sigma}(1+t)^{-1-\gamma}.$$

Thus estimates (3.6)–(3.7) are true with m replaced by $m+1$. Therefore by induction estimates (3.6)–(3.7) are fulfilled for all m.

In the same manner we prove the estimates

$$\|z_{m+1}-z_m\|_{\mathbf{X}}\leqslant\frac{1}{2}\|z_m-z_{m-1}\|_{\mathbf{X}},$$

where $\|\psi\|_{\mathbf{X}}\equiv\||\xi|^{-1}\psi\|_{\mathbf{L}^2}+\|\langle\xi\rangle\psi\|_{\mathbf{H}^{1,1}}$. Therefore there exists a unique solution of Eqs. (3.3)–(3.5) which obeys the estimates (3.6)–(3.7).

Returning to the function w we find that

$$\left\||\xi|^{-1}\widehat{w}\right\|_{\mathbf{L}^2}\leqslant C\varepsilon^{1+\sigma}(1+t)^{\frac{1}{4}-\gamma},$$
$$\|\widehat{w}\|_{\mathbf{L}^2}+\sqrt{t+1}\left\||\xi|\widehat{w}\right\|_{\mathbf{L}^2}\leqslant C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{4}-\gamma}.$$

The last estimate gives us

$$\|w\|_{\mathbf{L}^2}+\sqrt{t+1}\|w_x\|_{\mathbf{L}^2}\leqslant C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{4}-\gamma},$$

hence by the Sobolev imbedding inequality we get

$$\|w\|_{\mathbf{L}^\infty} \leqslant C\|w\|_{\mathbf{L}^2}^{\frac{1}{2}}\|w_x\|_{\mathbf{L}^2}^{\frac{1}{2}} \leqslant C\varepsilon^{1+\sigma}(1+t)^{-\frac{1}{2}-\gamma}.$$

This implies the estimate of the theorem. Theorem 1.1 is proved.

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References

[1] M. Escobedo, O. Kavian, Asymptotic behavior of positive solutions of a non-linear heat equation, Houston J. Math. 13 (4) (1987) 39–50.
[2] M. Escobedo, O. Kavian, H. Matano, Large time behavior of solutions of a dissipative nonlinear heat equation, Comm. Partial Differential Equations 20 (1995) 1427–1452.
[3] H. Fujita, On the blowing-up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. of Tokyo Sect. I 13 (1966) 109–124.
[4] A. Gmira, L. Veron, Large time behavior of the solutions of a semilinear parabolic equation in $\mathbf{R}^N$, J. Differential Equations 53 (1984) 258–276.
[5] K. Hayakawa, On non-existence of global solutions of some semi-linear parabolic differential equations, Proc. Japan Acad. 49 (1973) 503–505.
[6] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Large time behavior of solutions to the dissipative nonlinear Schrödinger equation, Proc. Roy. Soc. Edinburgh Sect. A 130 (2000) 1029–1043.
[7] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Landau–Ginzburg type equations in the subcritical case, Comm. Contemp. Math. 5 (1) (2003) 127–145.
[8] K. Kobayashi, T. Sirao, H. Tanaka, On the growing up problem for semi-linear heat equations, J. Math. Soc. Japan 29 (1977) 407–424.